MATHEMATICS

Chapter 13: LIMITS AND DERIVATIVES

LIMITS AND DERIVATIVES

Some useful results

1. $(a^2 - b^2) = (a + b)(a - b)$
2. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
3. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
4. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a + b)(a - b)(a^2 + b^2)$
5. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + .$
6. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots$
7. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \ldots$
8. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$
9. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$
10. $a^x = 1 + x(\log\ a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots$
11. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$
12. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$
13. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$
14. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
15. $\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$
16. $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
17. $\tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A \cos B}$
18. $\tan A - \tan B = \tan(A - B)\{1 + \tan A \tan B\}$
19. $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$
20. $\sin C - \sin D = 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}$

21. $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

22. $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

23. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

24. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

25. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Key Concepts

1. The expected value of the function as dictated by the points to the left of a point defines the left-handlimit of the function at that point. The limit $\lim_{x\to a^-} f(x)$ is the expected value of f at x = a given the values of f near x to the left of a.

2. The expected value of the function as dictated by the points to the right of point a defines the right- hand limit of the function at that point. The limit $\lim_{x\to a^+} f(x)$ is the expected value of f at x = a given the values of f near x to the left of a.

3. Let y = f(x) be a function. Suppose that a and L are numbers such that as x gets closer and closer to a,f(x) gets closer and closer to L we say that the limit of f(x) at x = a is L, i.e., $\lim_{x\to a} f(x) = L$.

4. Limit of a function at a point is the common value of the left- and right-hand limit if they coincide, i.e., $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

5. Real life examples of LHL and RHL

   a) If a car starts from rest and accelerates to 60 km/hr in 8 seconds, which means the initial speed of the car is 0 and it reaches 60 km 8 seconds after the start. On recording the speed of the car, we can see that this sequence of numbers is approaching 60 km in such a way that each member of the sequence is less than 60. This sequence illustrates the concept of approaching a number from the left of that number.

   b) Boiled milk which is at a temperature of 100 degrees is placed on a shelf; temperature goes ondropping till it reaches room temperature.

      As the time duration increases, temperature of milk, t, approaches room temperature say 30° .Thissequence illustrates the concept of approaching a number from the right of that number.

6. Let f and g be two functions such that both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exists then

   a. Limit of the sum of two functions is the sum of the limits of the functions,

   $$\text{i.e. } \lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

   b. Limit of the difference of two functions is the difference of the limits of the functions,

   $$\text{i.e. } \lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

   c. Limit of the product of two functions is the product of the limits of the functions,

   $$\text{i.e. } \lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x).$$

   d. Limit of the quotient of two functions is the quotient of the limits of the functions (whenever the denominator is non zero),

   $$\text{i.e. } \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

   e. $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ exists, then

   $$\lim_{x\to a}\left|1 \quad f(x)\right|^{\frac{1}{g(x)}} = e^{\lim\limits_{x\to a}\frac{f(x)}{g(x)}}$$

   f. If $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$ such that

   $\lim\limits_{x\to a}\left|f(x)-1\right|g(x)$ exists, then,

   $$\lim_{x\to a} f(x)^{g(x)} = e^{\lim\limits_{x\to a}\left|f(x)-1\right|g(x)}$$

7. For any positive integer n,

$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$$

8. Limit of a polynomial function can be computed using substitution or algebra of limits.
9. The following methods are used to evaluate algebraic limits:

   i. Direct substitution method

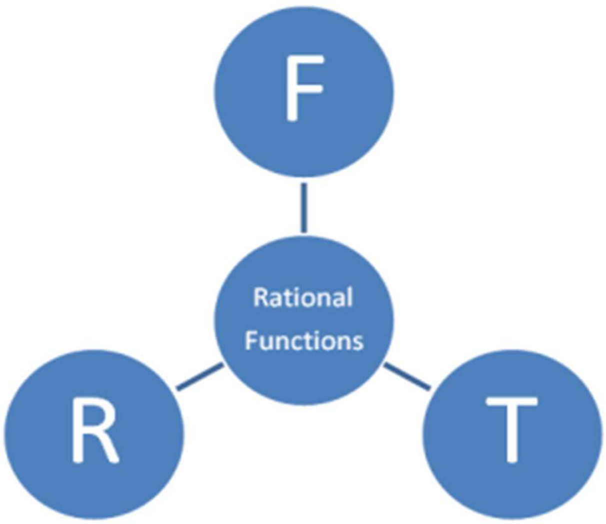
   ii. Factorization method

   iii. Rationalization method

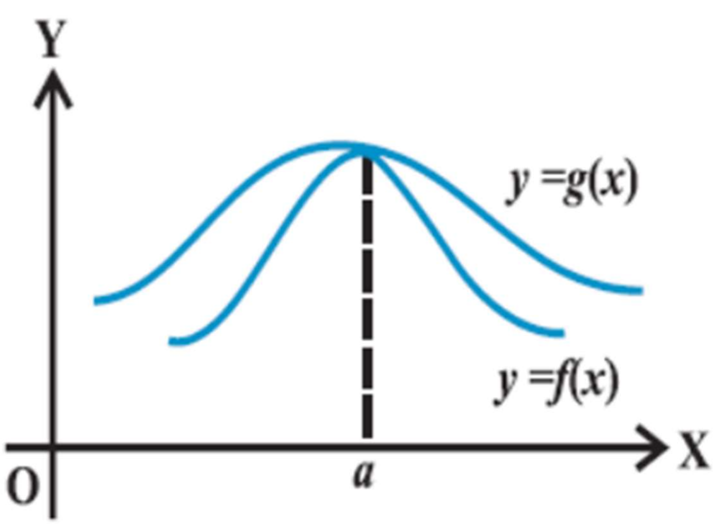
   iv. By using some standard limits

   v. Method of evaluation of algebraic limits at infinity

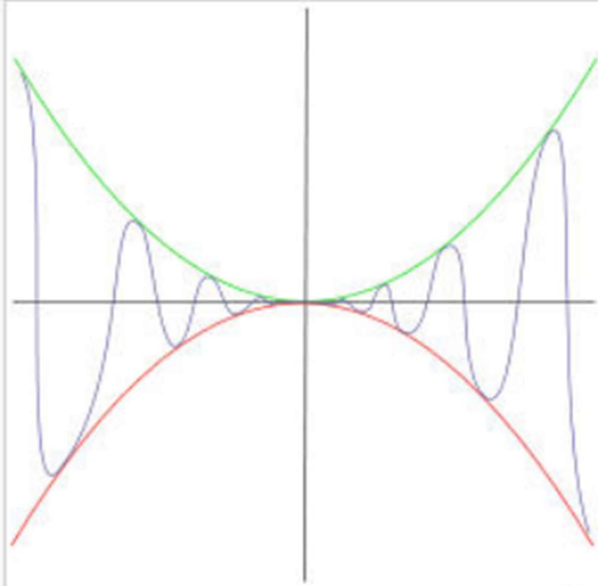
10. For computing the limit of a rational function when direct substitution fails, use factorisation ,rationalisation or the theorem.



11. Let f and g be two real valued functions with the same domain such that f(x) ≤ g(x) for all x in thedomain of definition. For some a, if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.



12.Let f, g and h be real functions such that f(x) ≤ g(x) ≤ h(x) for all x in the common domain of definition.For some real number a, if $\lim_{x\to a} f(x)$.= ℓ= $\lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = l$



An example of a squeezed function

13.Suppose f is a real valued function and a is a point in its domain of definition. The derivative of f at ais defined by

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Provided this limit exists and is finite. Derivative of f(x) at a is denoted by f'(a).

14.A function is differentiable in its domain if it is always possible to draw a unique tangent at every pointon the curve.

15.Finding the derivative of a function using definition of derivative is known as the first principle ofderivatives or ab-initio method.

16.Differentiation of a constant function is zero.

17. If f(x) is a differentiable function and 'c' is a constant, then $\frac{d}{dx}\left(c.f(x)\right) = c.\frac{d}{dx}f(x)$.

18.Let f and g be two functions such that their derivates are defined in a common domain. Then

i. Derivative of the sum of two functions is the sum of the derivatives of the functions.

$$\frac{d}{dx}[f(x)+g(x) = \frac{d}{dx}f(x)+\frac{d}{dx}g(x)$$

ii. Derivative of the difference of two functions is the difference of the derivatives of the functions.

$$\frac{d}{dx}[f(x)-g(x) = \frac{d}{dx}f(x)-\frac{d}{dx}g(x)$$

iii. Derivative of the product of two functions is given by the following products rule.

$$\frac{d}{dx}[f(x).g(x) = \frac{d}{dx}f(x).g(x)+f(x).\frac{d}{dx}g(x)$$

iv. Derivative of quotient of two functions is given by the following quotient rule(whenever the denominator is non-zero).

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}f(x).g(x)-f(x)\frac{d}{dx}g(x)}{(g(x))^2}$$

19. Generalization of the product rule: Let f(x), g(x) and h(x) be three differentiable functions. Then

$$\frac{d}{dx}[f(x)\cdot g(x)\cdot h(x)]$$
$$= \frac{d}{dx}[f(x)]\cdot g(x)\cdot h(x)+f(x)\cdot\frac{d}{dx}[g(x)]\cdot h(x)+f(x)\cdot g(x)\cdot\frac{d}{dx}[h(x)]$$

20. Derivative of $f(x) = x^n$ is nx^{n-1} for any positive integer n.

21. Let $f(x) = a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2x + a_1$.

Now, a_2x are all real numbers and $a_n \neq 0$. Then, the derivative function is given by

$$\frac{df(x)}{dx} = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2x + a_1.$$

22. For a function f and a real number a, $\lim_{x\to a} f(x)$ and f(a) may not be same (In fact, one may be defined and not the other one).

23. Standard derivatives

f(x)	f′(x)
sin x	cos x
cos x	−sin x
tan x	$\sec^2 x$
cot x	$-\text{cosec}^2 x$
sec x	sec x tan x
cosec x	−cosec x cot x
x^n	nx^{n-1}
c	0
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
e^x	e^x
$\frac{1}{\sqrt{x}}$	$\frac{-1}{2}x^{-\frac{3}{2}}$
$\frac{1}{x}$	$-\frac{1}{x^2}$
a^x	$a^x \log_e a$
$\log_e x$	$\frac{1}{x}$

24. The derivative is the instantaneous rate of change in the terms of Physics and is the slope of thetangent at a point.

25. A function is not differentiable at the points where it is not defined or at the points where the uniquetangent cannot be drawn.

26. Conisider that f’(x), $\frac{dy}{dx}$, $\frac{df(x)}{dx}$ and y’ are all different notations for the derivative with respect to x.

Key Formulae

1. $\lim_{x\to\infty} c = c$
2. $\lim_{x\to-\infty} c = c$
3. $\lim_{x\to\infty} \frac{c}{x^n} = 0, n > 0$
4. $\lim_{x\to-\infty} \frac{c}{x^n} = 0, n \in N$
5. $\lim_{x\to+\infty} x \to +\infty$
6. $\lim_{x\to-\infty} x \to -\infty$
7. $\lim_{x\to+\infty} x^2 \to +\infty$
8. $\lim_{x\to-\infty} x^2 \to \infty$
9. $\lim_{x\to\infty} e^x \to \infty$
10. $\lim_{x\to-\infty} e^{-x} \to \infty$
11. $\lim_{x\to\infty} e^{-x} \to 0$
12. $\lim_{x\to-\infty} e^x \to 0$
13. $\lim_{x\to\infty} a^x \to 0,$ if $|a| \quad 1$
14. $\lim_{x\to\infty} a^x \to \infty,$ if $|a| > 1$
15. $\lim_{x\to 0^+} \log_a x \to -\infty$ and $\lim_{x\to\infty} \log_a x \to -\infty,$ where a>1
16. $\lim_{x\to 0^+} \log_a x \to \infty$ and $\lim_{x\to\infty} \log_a x \to -\infty,$ where $0 < a < 1$
17. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$
18. $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$
19. $\lim_{x\to 0} \frac{e^x - 1}{x} = \log_e e = 1$
20. $\lim_{x\to 0} (1 \quad x)^{\frac{1}{x}} = e$
21. $\lim_{x\to 0} \left(1 + \frac{1}{x}\right)^x = e$
22. $\lim_{x\to 0} (1 + \lambda x)^{\frac{1}{x}} = e^{\lambda}$
23. $\lim_{x\to 0} \left(1 + \frac{\lambda}{x}\right)^x = e^{\lambda}$

24. $\lim_{x \to 0} \sin x = 0$
25. $\lim_{x \to 0} \cos x = 1$
26. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
27. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
28. $\lim_{x \to 0} \frac{\tan x}{x} = 1$
29. $\lim_{x \to a} \frac{\sin(x - a)}{(x - a)} = 1$
30. $\lim_{x \to a} \frac{\tan(x - a)}{(x - a)} = 1$

Steps for finding the left-hand limit

1. **Step 1:** Get the function $\lim_{x \to a^-} f(x)$
2. **Step 2:** Substitute x = a - h and replace $\mathbf{x \to a^- \text{ by } h \to 0}$ to get $\lim_{h \to 0} f(a - h)$
3. **Step 3:** Using appropriate formula simplify the given function.
4. **Step 4:** The final value is the left-hand limit of the function at x = a.

Steps for finding the right-hand limit

1. **Step 1:** Get the function $\lim_{x \to a^+} f(x)$
2. **Step 2:** Substitute x = a - h and replace $\mathbf{x \to a^+ \text{ by } h \to 0}$ to get $\lim_{h \to 0} f(a + h)$
3. **Step 3:** Using appropriate formula simplify the given function.
4. **Step 4:** The final value is the left-hand limit of the function at x = a.

Steps for factorisation method

1. **Step 1:** Get the function $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$
2. **Step 2:** Factorize f x and g x.
3. **Step 3:** Cancel out the common factors.

4. **Step 4:** Use the direct substitution method to find the final limit**.**

Steps for rationalisation method

1. When the numerator or denominator or both involve expression takes the form $\frac{\mathbf{0}}{\mathbf{0}}, \frac{\infty}{\infty}$ we can use this method.

In this method, factor out the numerator and the denominator separately and cancel the common factor

Example:

Evaluate $\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$:

Solution:

At x=0, $\frac{\sqrt{2+x} - \sqrt{2}}{} \to \frac{0}{0}$

Thus, rationalising the numerator, we have,

$$\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{\left(\sqrt{2+x} - \sqrt{2}\right)\left(\sqrt{2+x} + \sqrt{2}\right)}{x\left(\sqrt{2+x} + \sqrt{2}\right)}$$

$$= \lim_{x \to 0} \frac{2+x-2}{x\left(\sqrt{2+x} + \sqrt{2}\right)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{2+x} + \sqrt{2}}$$

$$= \frac{1}{2\sqrt{}}$$

MIND MAP : LEARNING MADE SIMPLE CHAPTER - 13

Limits and Derivatives

Limit

- We say $\lim_{x \to a^-} f(x)$ is the expected value of f at $x=a$ given that the values of f near x to the left of a. This value is called the left hand limit of f at 'a'
- We say $\lim_{x \to a^+} f(x)$ is the expected value of f at $x=a$ given that the values of f near x to the right of a. This value is called the right hand limit of f 'at a'.
- If the right and left hand limits coincide, we call that common value as the limit of $f(x)$ at $x=a$ and denoted it by $\lim_{x \to a} f(x)$.

Eg: Find limit of function $f(x)=(x-1)^2$ at $x=1$.

Sol: For $f(x)=(x-1)^2$

Left hand limit (LHL) (at x=1) $= \lim_{x \to 1^-}(x-1)^2 = 0$

and Right hand limit (RHL) (at x=1) $= \lim_{x \to 1^+}(x-1)^2 = 0$

$\therefore$ LHL = RHL

$\lim_{x \to 1}(x-1)^2 = 0$

Derivative of a function at 'a'

The derivative of a function f at a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

Eg: Find derivative of $f(x) = \frac{1}{x}$.

Sol: We have $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{-h}{x(x+h)}\right] = \frac{-1}{x^2}$$

Derivative of polynomial function

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + - - + a_1 x + a_0$ be a polynomial function, where a_i s are all real numbers and $a_n \neq 0$. Then the derivative function is given by

$$\frac{d f(x)}{dx} = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + - - + 2a_2 x + a_1$$

Algebra of derivatives of functions

For functions u and v the following holds:

- $(u \pm v)' = u' \pm v'$
- $(uv)' = u'v + v'u$
- $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$ provided all are defined and $v \neq 0$

Here, $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$

Some Standard Derivatives

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$

Some Standard Limits

- $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
- $\lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Algebra of Limits

For functions f and g the following holds:

- $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $\lim_{x \to a}[f(x).g(x)] = \lim_{x \to a} f(x).\lim_{x \to a} g(x)$
- $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, $\therefore \lim_{x \to a} g(x) \neq 0$

Important Questions

Multiple Choice questions-

Question 1. The expansion of log(1 – x) is:

(a) $x – x^2/2 + x^3/3 –$ ……..

(b) $x + x^2/2 + x^3/3 +$ ……..

(c) $-x + x^2/2 – x^3/3 +$ ……..

(d) $-x – x^2/2 – x^3/3 –$ ……..

Question 2. The value of $\lim_{x \to a} (a \times \sin x – x \times \sin a)/(ax^2 – xa^2)$ is

(a) = $(a \times \cos a + \sin a)/a^2$

(b) = $(a \times \cos a – \sin a)/a^2$

(c) = $(a \times \cos a + \sin a)/a$

(d) = $(a \times \cos a – \sin a)/a$

Question 3. $\lim_{x \to -1} [1 + x + x^2 + \ldots\ldots + x^{10}]$ is

(a) 0

(b) 1

(c) -1

(d) 2

Question 4. The value of the limit $\lim_{x \to 0} \{\log(1 + ax)\}/x$ is

(a) 0

(b) 1

(c) a

(d) 1/a

Question 5. The value of the limit $\lim_{x \to 0} (\cos x)\cot^{2x}$ is

(a) 1

(b) e

(c) $e^{1/2}$

(d) $e^{-1/2}$

Question 6. Then value of $\lim_{x \to 1} (1 + \log x – x)\}/(1 – 2x + x^2)$ is

(a) 0

(b) 1

(c) 1/2

(d) -1/2

Question 7. The value of $\lim_{y\to0}$ {(x + y) × sec (x + y) – x × sec x}/y is

(a) x × tan x × sec x

(b) x × tan x × sec x + x × sec x

(c) tan x × sec x + sec x

(d) x × tan x × sec x + sec x

Question 8. $\text{Lim}_{x\to0}$ $(ex^2 - \cos x)/x^2$ is equals to

(a) 0

(b) 1

(c) 2/3

(d) 3/2

Question 9. The expansion of a^x is:

(a) $a^x = 1 + x/1! \times (\log a) + x^2/2! \times (\log a)^2 + x^3/3! \times (\log a)^3 +$ ……….

(b) $a^x = 1 - x/1! \times (\log a) + x^2/2! \times (\log a)^2 - x^3/3! \times (\log a)^3 +$ ……….

(c) $a^x = 1 + x/1 \times (\log a) + x^2/2 \times (\log a)^2 + x^3/3 \times (\log a)^3 +$ ……….

(d) $a^x = 1 - x/1 \times (\log a) + x^2/2 \times (\log a)^2 - x^3/3 \times (\log a)^3 +$ ……….

Question 10. The value of the limit $\text{Lim}_{n\to0} (1 + an)^{b/n}$ is:

(a) e^a

(b) e^b

(c) e^{ab}

(d) $e^{a/b}$

Very Short Questions:

1. Evaluate $\lim_{x\to3}\left[\frac{x^2-9}{x-3}\right]$
2. Evaluate $\lim_{x\to0}\frac{\sin 3x}{5x}$
3. Find derivative of 2x.
4. Find derivative of $\sqrt{\sin 2x}$
5. Evaluate $\lim_{x\to0}\frac{\sin^2 4x}{x^2}$
6. What is the value of $\lim_{x\to a}\left(\frac{x^2-a^n}{x-a}\right)$

7. Differentiate $\frac{2x}{x}$

8. If y = $e^{\sin x}$ Find $\frac{dy}{dx}$

9. Evaluate $\lim_{x\to 1}\frac{x^{15}-1}{x^{10}-1}$

10. Differentiate x sin x with respect to x.

Short Questions:

1. Prove that $\lim_{x\to 0}\left(\frac{e^{x}-1}{x}\right) = 1$

2. Evaluate $\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)} = 1$

3. Evaluate $\lim_{x\to 0}\frac{x\ tan\ 4x}{1-\cos 4x}$

4. It $y = \frac{(1-\tan\ \ )}{(1+\tan\ \ )}$. Show that $\frac{dy}{dx} = \frac{-2}{(1+\sin 2x)}$

5. Differentiate $e^{\sqrt{\cot x}}$

Long Questions:

1. Differentiate tan x from first principle.

2. Differentiate $(x + 4)^6$ From first principle.

3. Find derivative of cosec x by first principle.

4. Find the derivatives of the following fuchsias:

$(i)\ \left(x-\frac{1}{x}\right)^3 \quad (ii)\ \frac{(3x+1)\left(2\sqrt{x-1}\right)}{\sqrt{x}}$

5. Find the derivative of sin (x + 1) with respect to x from first principle.

Assertion Reason Questions:

1. In each of the following questions, a statement of Assertion is given followed by a corresponding statement of Reason just below it. Of the statements, mark the correct answer as.

Assertion (A) $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}$ is equal to 1, where $a+b+c\neq 0$.

Reason (R) $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$ is equal to $\frac{1}{4}$.

(i) Both assertion and reason are true and reason is the correct explanation of assertion.

(ii) Both assertion and reason are true but reason is not the correct explanation of assertion.

(iii) Assertion is true but reason is false.

(iv) Assertion is false but reason is true.

2. In each of the following questions, a statement of Assertion is given followed by a corresponding statement of Reason just below it. Of the statements, mark the correct answer as.

Assertion (A) $\lim_{x \to 0} \frac{\sin ax}{bx}$ is equal to $\frac{a}{b}$.

Reason (R) $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

(i) Both assertion and reason are true and reason is the correct explanation of assertion.

(ii) Both assertion and reason are true but reason is not the correct explanation of assertion.

(iii) Assertion is true but reason is false.

(iv) Assertion is false but reason is true.

Answer Key:

MCQ:

1. (d) $-x - x^2/2 - x^3/3 - \ldots\ldots$

2. (b) $= (a \times \cos a - \sin a)/a^2$

3. (b) 1

4. (c) a

5. (d) $e^{-1/2}$

6. (d) -1/2

7. (d) $x \times \tan x \times \sec x + \sec x$

8. (d) 3/2

9. (a) $a^x = 1 + x/1! \times (\log a) + x^2/2! \times (\log a)^2 + x^3/3! \times (\log a)^3 + \ldots\ldots\ldots$

10. (c) e^{ab}

Very Short Answer:

1.

$$\lim_{x\to 3}\frac{x^2-9}{x-3}\ \frac{0}{0}\ form$$

$$\lim_{x\to 3}\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}=3+3=6$$

2.

$$\lim_{x\to 0}\frac{\sin 3x}{5x}$$

$$=\lim_{3x\to 0}\frac{\sin 3x}{3x}\times\frac{3}{5}$$

$$=1\times\frac{3}{5}=\frac{3}{5}\left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

3. Let y = 2^x

$$\frac{dy}{dx}=\frac{d}{dx}2=2^x 10g2$$

4.

$$\frac{d}{dx}\sqrt{\sin 2x}=\frac{1}{2\sqrt{\sin 2x}}\frac{d}{dx}\sin 2x$$

$$=\frac{1}{\cancel{2}\sqrt{\sin 2x}}\times\cancel{2}\cos 2x$$

$$=\frac{\cos 2x}{\sqrt{\cos 2x}}$$

5.

$$\lim_{x\to 0}\frac{\sin^2 4x}{x^2 4^2}\times 4^2=\lim_{4x\to 0}\left(\frac{\sin 4x}{4x}\right)^2\times 16$$

$$=1\times 16=16$$

6.

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=1$$

7.

$$\frac{d}{dx}\frac{2^x}{x}=\frac{x\frac{d}{dx}2^x-2^x\frac{d}{dx}x}{x^2}$$

$$=\frac{x\times 2^x 10g2-2^x\times 1}{x^2}$$

$$= 2x\frac{[x+10g2-1]}{x^2}$$

8.

$$y = e^{\sin x}$$

$$\frac{dy}{dx} = \frac{d}{dx}e^{\sin x}$$

$$= e^{\sin x} \times \cos x = \cos x e^{\sin x}$$

9.

$$\lim_{x\to 1}\frac{x^{15}-1}{x^{10}-1}$$

$$= \frac{\lim_{x\to 1}\frac{x^{15}-1^{15}}{x-1}}{\lim_{x\to 1}\frac{x^{10}-1^{10}}{x-1}} = \frac{15\times 1^{14}}{10\times 1^{9}}$$

$$= \frac{15}{10} = \frac{3}{2}$$

10.

$$\frac{d}{dx}x\sin x = x\cos x + \sin x.1$$

$$= x\cos x + \sin x$$

Short Answer:

1. We have

$$\lim_{x\to 0}\frac{e^x-1}{x}$$

$$\lim_{x\to 0}\left\{\frac{\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+---\right]-1}{x}\right\}\left[\because e^x = 1+x+\frac{x^2}{2!}+---\right]$$

$$\lim_{x\to 0}\left\{\frac{x+\frac{x^2}{2!}+\frac{x^3}{3!}+---}{x}\right\}$$

$$\lim_{x\to 0}x\left\{\frac{1+\frac{x}{2!}+\frac{x^2}{3!}+---}{x}\right\}$$

= 1 + 0 = 1

2.

$$\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}$$

$$=\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$$

$$\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}\times\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}$$

$$\lim_{x\to 1}\frac{(2x-3)(x-1)}{(2x+3)(x-1)(\sqrt{x}+1)}$$

$$\lim_{x\to 1}\frac{(2x-3)}{(2x+3)(\sqrt{x}+1)}=\frac{2\times 1-3}{(2\times 1+3)(\sqrt{1}+1)}$$

$$=\frac{-1}{10}$$

3.

$$\lim_{x\to 0}\frac{x\tan 4x}{1-cas4x}$$

$$=\lim_{x\to 0}\frac{x\sin 4x}{\cos 4x\left[2\sin^2 2x\right]}$$

$$=\lim_{x\to 0}\frac{2x\sin 2x\cos 2x}{\cos 4x\left(2\sin^2 2x\right)}$$

$$=\lim_{x\to 0}\left(\frac{\cos 2x}{\cos 4x}.\frac{2x}{\sin 2x}\times\frac{1}{2}\right)$$

$$=\frac{1}{2}\frac{\lim_{2x\to 0}\cos 2x}{\lim_{4x\to 0}\cos 4x}\times\lim_{2x\to 0}\left(\frac{2x}{\sin 2x}\right)=\frac{1}{2}\times 1=\frac{1}{2}$$

4.

$$y=\frac{(1-\tan x)}{(1+\tan x)}$$

$$\frac{dy}{dx}=\frac{(1+\tan x)\frac{d}{dx}(1-\tan x)-(1-\tan x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$=\frac{(1+\tan x)(-\sec^2 x)-(1-\tan x)\sec^2 x}{(1+\tan x)^2}$$

$$=\frac{-\sec^2 x-\tan x\sec^2 x-\sec^2+\tan x\sec^2 x}{(1+\tan x)^2}$$

$$=\frac{-2\sec^2 x}{(1+\tan x)^2}=\frac{-2}{\cos^2 x\left[1+\frac{sicx}{\cos x}\right]^2}$$

$$=\frac{-2}{\cos^2 x\left[\frac{\cos x+\sin x}{\cos^2 x}\right]^2}$$

$$=\frac{-2}{\cos^2 x+\sin^2 x+2\sin x\cos x}=\frac{-2}{1+\sin^2 x}$$

$$\therefore \frac{dy}{dx}=\frac{-2}{1+\sin 2x}$$

Hence Proved.

5.

Let $y=e^{\sqrt{\cot x}}$

$$\frac{dy}{dx}=\frac{d}{dx}e^{\sqrt{\cot x}} = e^{\sqrt{\cot x}}\frac{d}{dx}\sqrt{\cot x}$$

$$=e^{\sqrt{\cot x}}\times\frac{1}{2\sqrt{\cot x}}.\frac{d}{dx}\cot x$$

$$=\frac{e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}-\cos ec^2 x$$

$$=\frac{-\cos cec^2 e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}$$

Long Answer:

1.

$$f(x)=\tan x$$

$$f(x+h)=\tan(x+h)$$

$$f^1(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$= \lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0}\frac{\sin[x+h-x]}{h\cos(x+h)\cos x}\left[\because \sin(A-B)= \sin A\cos B-\cos A\sin B\right]$$

$$= \lim_{h\to 0}\frac{\sinh}{h\cos(x+h)\cos x}$$

$$= \frac{\lim_{h\to 0}\frac{\sinh}{h}}{\lim_{h\to 0}\cos(x+h)\cos x}=\frac{1}{\cos x.\cos x}\left[\because \lim_{h\to 0}\frac{\sinh}{h}=1\right]$$

$$= \frac{1}{\cos^2 x}=\sec^2 x$$

2.

let $f(x)=(x+4)^6$

$f(x+h)=(x+h+4)^6$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h+4)^6-(x+4)^6}{h}$$

$$= \lim_{(x+h+4)\to(x+4)}\frac{(x+h+4)^6-(x+4)^6}{(x+h+4)-(x-4)}$$

$$= 6(x+4)^{(6-1)}\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$= 6(x+4)^5$$

3.

proof let $f(x)=\operatorname{cosec} x$

By def, $f(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(h)}{h}$

$$= \underset{h\to 0}{Lt}\frac{\operatorname{cosec}(x+h)-\operatorname{cosec} x}{h}$$

$$= \underset{h\to 0}{Lt}\frac{\frac{1}{\sin(x+h)}-\frac{1}{\sin x}}{h} = \underset{h\to 0}{Lt}\frac{\sin x-\sin(x+h)}{h\sin(x+h)\sin x}$$

$$= \underset{h\to 0}{Lt}\frac{2\cos\frac{x+x+h}{2}\sin\frac{x-x+h}{2}}{h\sin(x+h)\sin x}$$

$$= \underset{h\to 0}{Lt}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h\sin(x+h)\sin x}$$

$$= \frac{\underset{\frac{h}{2}\to 0}{Lt}\cos\left(x+\frac{h}{2}\right)}{\cos x.\underset{h\to 0}{L}+\sin(x+h)}.\underset{\frac{h}{2}\to 0}{Lt}\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$$= -\frac{\cos x}{\sin x.\sin x}.1 = -\cos ecx\cot x$$

4.

(i) $\text{let } f(x) = \left(x-\frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3x + \frac{1}{x}\left(x-\frac{1}{x}\right)$

$= x^3 - x^{-3} - 3x + 3x^{-1}.d.ff\ wr.t4, we$ get

$f(x) = 3\times x^2 - (-3)x^{-4} - 3\times 1 + 3\times(-1)x^{-2}$

$= 3x^2 + \frac{3}{x^4} - 3 - \frac{3}{x^2}.$

(ii) $\text{let } f(x) = \frac{(3x+1)(2\sqrt{x}-1)}{\sqrt{x}} = \frac{6x^{\frac{3}{2}} - 3x + 2\sqrt{x} - 1}{\sqrt{x}}$

$= 6x - 3x^{\frac{1}{2}} + 2 - x^{-\frac{1}{2}}, d : ff$ w.r.t. x.weget

$f(x) = 6\times 1 - 3\times\frac{1}{2}\times x^{-\frac{1}{2}} + 0 - \left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$

$= 6 - \frac{3}{2\sqrt{x}} + \frac{1}{2, x^{\frac{3}{2}}}.$

5.

let $f(x) = \sin(x+1)$

$f(x+h) = \sin(x+h+1)$

$$f^1(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h+1) - \sin(x+1)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left[\frac{x+h+1+x+1}{2}\right]\sin\left[\frac{x+h+1-x-1}{2}\right]}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left[x+1+\frac{h}{2}\right]\sin\frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \cancel{2}\cos\left(x+1+\frac{h}{2}\right) \times \lim_{h \to 0} \frac{\sin\frac{h}{2}}{\cancel{2}\frac{h}{2}}$$

$$= \cos(x+1) \times 1 = \cos(x+1)$$

Assertion Reason Answer:

1. (iii) Assertion is true but reason is false.

2. (i) Both assertion and reason are true and reason is the correct explanation of assertion.